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Markov Semigroups of Operators and Transition Functions

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Abstract

The subject of the present work deals with the concept of markov semigroups and their characterization by means of the notion of transition function also called markovian kernel. In this context the Hille-Yosida theory of semigroups of operators will be useful. The operators we consider here are acting on the Banach space *C (X)* of real continuous functions on the metric compact space *X*, with the uniform norm $\|f\| = \sup\{|f(x)|, x \in X\}$.

Keywords: Markov semigroups; Transition functions; Pre-generators; Markovian Generators

Introduction

We consider a special class of semigroups of operators S_t, $t \geq 0$, called Markov semigroups and acting on the Banach space *C (X)* of all real continuous functions on compact metric space *X*. Such semigroups are closely related to the notion of transition function also called markovian kernel. Under some regularity conditions we will see that a transition function gives rise to a Markov semigroup. By the way we need some convergence results about real measures. To this end we endow the space *X* with its Borel $\sigma-$ field $\mathcal{B}$; that is the $\sigma-$ field generated by the open sets of *X* or by all real continuous functions on *X*, which is the same. It is known that the strong dual of *C(X)* is the Banach *M* space of all real measures μ on the measurable space *(X, B)*, by the representation Theorem of Riesz. We will be concerned by the subset of $M_1 \subset M$ all probabilty measures equiped with the weak$*$ topology which gives the convergence we need: a sequence $(\mu_n) \subset M_1$ converges to $\mu \in M_1$ if $\int_X f d\mu_n$ converges to $\int_X f d\mu_n$ for all $f \in C(X)$.

We start with the definition of Markov semigroups on the Banach space *C(X)* and their continuity properties, giving some familiar examples. Then we introduce the notion of transition functions focusing attention on their relation to Markov semigroups. These considerations lead to the notion of stochasticcontinuity and the special class of Feller transition function. One important fact that manages these concepts is the Chapmann-Kolmogorov equation that must be satisfied by a transition function.

The last part of this paper is intended to the generation of Markov semigroups and an application to a differential equation.

Markov Semigroups

Let *X* be a compact metric space endowed with its Borel $\sigma-$ field $\mathcal{B}$; that is the $\sigma-$ field generated by the open sets of *X* or by all real continuous functions on *X*, which is the same. We consider the Banach space *C(X)* of all real continuous functions on *X*.

Definition

A family of operators $S_t, t \geq 0$, on the Banach space *C(X)* is called a Markov semigroup if:

(1) $S_{t+s} = S_t S_s, \forall t, s \geq 0$ ($S_0 = I$, the identity operator)

(2) $\lim_{t \to 0_+} S_t f = f, \forall \in C(X)$

(3) $S_t f \geq 0$ for $f \geq 0$

(4) $S_t 1 = 1, \forall t \geq 0$, (1 is the constant function equal 1).

Remark

(a) conditions (3), (4) imply that $S_t, t \geq 0$ is a contractions semigroup: indeed if $f \leq g$ we have by (3) $S_t f \leq S_t g$, so since for $f \in C(X) - \|f\| \leq f \leq \|f\|$ we deduce $-\|f\| S_t 1 \leq S_t f \leq \|f\| S_t 1$ and then $|S_t f| \leq \|f\|$, whence $|S_t f| \leq \|f\|$; therefore $\|S_t\| \leq 1$ and $\|S_t\| = 1$ because $S_t 1 = 1$.

(b) condition (2) means $\lim_{t \to 0_+} \|S_t f - f\| = 0$ and implies easily the uniform continuity of the function $t \to S_t f$ for $t > 0$, since we have:

$$\in \to 0_+ \Rightarrow \|S_{t+\in} f - S_t f\| = \|S_t (S_\in f - f)\| \leq \|S_\in f - f\| \to 0$$

$$\in \to 0_- \Rightarrow \|S_{t+\in} f - S_t f\| = \|S_{t+\in} (f - S_{-\in} f)\| \leq \|f - S_{-\in} f\| \to 0$$

Example

Let $G : C(X) \to C(X)$ be a bounded operator such that: $Gf \geq 0$ for $f \geq 0$ and $G1 = 1$ Define S_t, $t \geq 0$, on the Banach space $C(X)$ by $f \in C(X)$, $S_t f = e^{-t} . \sum_{n \geq 0} \frac{t^n}{n!} . G^n f$, $t \geq 0$

This is in fact the exponential semigroup $\exp(t[G-1])$. The conditions on G imply that S_t is Markovian.

Transition Functions

Definition

A transition function is a function $p_t(x,A)$ with $t \geq 0$, $x \in X$, $A \in B$ such that:

1. The set function $p_t(x,.)$ is a probability measure on B for any $t \geq 0, x \in X$
2. The function $p_t(.,A)$ is measurable for any $t \geq 0$, $A \in B$

3. **Chapmann-Kolmogorov equation:**
$p_{t+s}(x, A) = \int_X P_t(x, dy) . P_s(y, A)$, for any $t, s \geq 0, x \in X, A \in B$

4. $P_0(x,.) = \delta_x(.)$, the Dirac function at x

In several applications the transition function has the following interpretation: let us observe the random evolution of some system in the state space X suppose the system has state x at time $t = 0$, then for $t > 0$, $p_t(x,A)$ is the Probability that the system will jump to a state in A at time t.

It is known that a transition function, under some mild regularity conditions, can be used to construct a stochastic process with values in X and having interesting path continuity properties [1,2]. Now we consider the important class of Feller transition functions:

Definition

A transition function $p_t(x, A)$ has the Feller property if the function $x \to P_t(x,.)$ from X into the set M_I of probability measures on X is weak$*$ continuous that is, if for each t

$$x_n \to\to x \Rightarrow \int_X P_t(x_n, dy) . f(y) \to \int_X P_t(x, dy) . f(y), n \to \infty \text{ for all } f \in c(X)$$

Definition

A transition function $p_t(x, A)$ is uniformly stochastically continuous if it satisfies the condition:

(C) for each $\in > 0$, $\lim_{t \to 0_+} . P_t(x, U_\in(x)) = 1$ uniformly with respect to $x \in X$

$U_\in(x)$ being the open ball centered at x with radius $\in$.

(C) is equivalent to $\lim_{t \to 0_+} . \sup_{x \in X} . [1 - P_t(x, U_\in(x))] = 0$ for each $\in > 0$.

Theorem

Let $p_t(x, A)$ be a Feller transition function on X satisfying the condition *(C)* of the Definition 5.3, and put for any $f \in C(X), S_t f(x) = \int_X P_t(x,dy).f(y)$, then $\{S_t, t \geq 0\}$ is a Markov semigroup on $C(X)$.

Proof

We have to check the conditions of Definition 5.1 for $\{S_t, t \geq 0\}$ first by the Feller property of $p_t(x, A)$ we have $S_t f \in C(X)$. By Chapmann-Kolmogorov equation we can write

$$\int_X P_{t+s}(x,dy).I_A(y) = \int_X P_t(x,dz).\int_X P_s(z,dy).I_A(y)$$

This relation can be extended to simple functions by linearity, and, by monotone convergence, it will be satisfied by any bounded measurable function. So in particular for $f \in C(X)$ we have $\int_X P_{t+s}(x,dy).f(y) = \int_X P_t(x,dz).\int_X P_s(z,dy).f(y)$, that is $S_{t+s} f = S_t(S_s f)$, and this gives the validity (1) Definition 4.1; on the other hand since $P_0(x,.) = \delta_x(.)$ we have $S_0 f(x) = \int_X \delta_x(dy).f(y) = f(x)$ Conditions (3), (4) are trivial; let us show condition (2) that is $\lim_{t \to 0_+} S_t f = f$, $f \in C(X)$: since X is compact f is uniformly continuous [3,4], so for $\in > 0$ there is $\eta = \eta_\in > 0$ such that if d is the metric of X

$$x, y \in X, d(x,y) < \eta \Rightarrow |f(x) - f(y)| < \in$$

Let $U_\eta(x)$ be the open ball centered at x with radius η we have

$$|S_t f(x) - f(x)| = \left|\int_X P_t(x,y).(f(y) - f(x))\right| \leq \int_X P_t(x,dy).|f(y) - f(x)|$$

$$= \int_{u_\eta(x)} P_t(x,dy).|f(y) - f(x)| + \int_{X \setminus U_\eta(x)} P_t(x,dy).|f(y) - f(x)|$$

$$\leq \sup_{y \in U_\eta(x)} .|f(y) - f(x)| + 2\|f\|.[1 - P_t(x, U_\eta(x))$$

$$\leq \in + 2\|f\|.\sup_{x \in X}.[1 - P_t(x, U_\eta(x))$$

We deduce that $\|S_t f - f\| \leq \in + 2\|f\|.\sup_{x \in X}.\left[1 - P_t(x, U\eta(x))\right]$ letting t goes to 0 we get $\sup_{x \in X}.\left[1 - P_t(x, U\eta(x))\right] \to 0$ By the stochastic continuity of $p_t(x, A)$ and then

$\lim_{t \to 0}.\|S_t f - f\| \leq \in \forall_\in > 0$ So we have $\lim_{t \to 0}.\|S_t f - f\| = 0$ we just proved that any Feller transition function satisfying condition *(C)* generates a Markov semigroup. Conversely one can prove

Theorem

Let $\{S_t, t \geq 0\}$ be a Markov semigroup on $C(X)$. Then there is a unique Feller transition function satisfying condition *(C)* such that:

$$S_t f(x) = \int_X P_t(x,dy).f(y), \forall f \in C(X)$$

Markovian Generators

Definition

Let Ω be a linear operator on $C(X)$ with domain $D(\Omega)$. We say that Ω is a markovian pre-generator if:

(a) $1 \in D(\Omega)$ and $\Omega 1 = 0$ (1 is the constant function on X equal to 1)
(b) $D(\Omega)$ is dense in $C(X)$
(c) If $f \in D(\Omega), \lambda \geq 0$ and $f - \lambda \Omega f = g$ then $\min_x.f(x) \geq .\min_x.g(x)$

Proposition

Let Ω be a markovian pre-generator, then we have:

$(c')\, \forall\lambda \geq 0 \|f\| \leq \|f - \lambda\Omega f\|, \forall f \in D(\Omega)$ in this case we say that is dissipative.

Proof

We apply condition *(C)* of Definition 6.1 to *f* and −*f* in $D(\Omega)$ with $f - \lambda\Omega f = g$.

We get $\min_x . f(x) \geq . \min_x . g(x)$ and $\min_x . f(x) \geq . \min_x . -g(x)$ and deduce:

$$-\|g\| \leq . \min_x . g(x) \leq \min_x . f(x) \leq f(x) \leq \max_x . -f(x) \leq . \max_x . g(x) \leq \|g\|$$

whence $\forall x \in X, -\|g\| \leq f(x) \leq \|g\|$ and then $\|f\| \leq \|g\|$ since $f - \lambda\Omega f = g$, we get (c').

Remark

Let us observe that in the relation $f - \lambda\Omega f = g$ the function *g* determines *f* uniquely by (c'); indeed if $f_1 - \lambda\Omega f_1 = f_2 - \lambda\Omega f_2$ then $(f_1 - f_2) - \lambda\Omega(f_1 - f_2) = 0$ so $\|f_1 - f_2\| \leq 0$ whence $f_1 = f_2$ since in this case we take $g = 0$.

In order to check condition *(C)* of Definition 6.1 we frequently use the following proposition:

Proposition

Let Ω be satisfying the following principle:

$$f \in D(\Omega), \text{ and } f(z) = \min_x . f(x) \Rightarrow \Omega f(z) \geq 0$$

then satisfies condition *(C)* of Definition 6.1, so Ω is dissipative.

Proof

Let $z \in X$ with $f(z) = \min_x . f(x)$ such *z* exists since *f* is continuous on a compact space *X*, so for $\lambda \geq 0$ we have $-\lambda\Omega f(z) \leq 0$ and $f(z) - \lambda\Omega f(z) \leq f(z) = \min_x . f(x)$, therefore $\min_x . f(x) \geq f(z) - \lambda\Omega f(z) = g(z) \geq \min_x . g(x)$ which is condition *(C)*.

Example

Let *G* be a linear operator on *C*(*X*) such that *G*(1) = 1 and G(*f*)≥0 if *f*≥0, then Ω=*G-1* is a markovian pre-generator. We use proposition 6.3 to check condition (*C*) if $f(z) = \min_x . f(x)$ then *f-f(z)*≥0 and so $G(f - f(z)) = Gf - f(z) \geq 0$ this implies that $\Omega f(z) = Gf(z) - f(z) \geq 0$.

Definition

(1) Let Ω be a linear operator on *C*(*X*) with domain *D*(Ω). We say that Ω is closed if its graph $\Gamma = \{(f, \Omega f), f \in D(\Omega)\}$ is closed in the product space *C*(*X*)× *C*(*X*). In other words, Ω is closed if for any sequence $(f_n) \subset D(\Omega)$ such that $f_n \to f$ and $\Omega f_n \to g$, we have $f \in D(\Omega)$ and Ω*f*=*g*.

(2) A linear operator Ω_1 on *C*(*X*) with domain $D(\Omega_1)$ and graph Γ_1 is an extension of the operator Ω,*D*(Ω) with graph Γ if $\Gamma \subset \Gamma_1$ that is $D(\Omega) \subset D(\Omega_1)$ and $\Omega_1 f = \Omega f, \forall f \in D(\Omega)$.

(3) If the closure $\overline{\Gamma}$ of a graph Γ of some operator Ω,*D*(Ω) is the graph of an operator $\overline{\Omega}, D(\overline{\Omega})$ we say that $\overline{\Omega}$ is the closure of Ω it is in fact the minimal closed extension of Ω.

Remark

The closure $\overline{}$ of a graph Γ defines a linear operator iff:
$\forall g \neq 0, (0, g) \notin \overline{\Gamma}$ For markovian pre-generators the situation is given by:

Theorem

Let Ω be a markovian pregenerator, then Ω has a closure $\overline{\Omega}$ which is also a markovian pregenerator.

Proof

We prove that $(0.h) \in \overline{\Gamma} \Rightarrow h = 0$

Let $(f_n) \subset D(\Omega)$ with $f_n \to 0, \Omega f_n \to h$ Since by Proposition 6.2, Ω is dissipative we have for $g \in D(\Omega)$ and λ≥0

$$\left\|\left(I-\lambda\Omega\right)\left(f_n+\lambda_g\right)\right\| \geq \left\|\left(f_n+\lambda_g\right)\right\|, \lambda \geq 0$$

But $\left(I-\lambda\Omega\right)\left(f_n+\lambda_g\right)=f_n-\lambda\Omega f_n+\lambda g-\lambda^2\Omega g \to \lambda g-\lambda^2\Omega g-\lambda h, n\to\infty$ and $\left\|f_n+\lambda g\right\| \to \left\|\lambda_g\right\|$ we deduce the inequality $\left\|\lambda g-\lambda^2\Omega g-\lambda h\right\| \geq \left\|\lambda g\right\|$, valid for all λ>0 and all $g\in D(\Omega)$, so we get $\lambda^{-1}\left\|\lambda g-\lambda^2\Omega g-\lambda h\right\| \geq \lambda^{-1}\left\|\lambda g\right\|$ that is $\left\|g-\lambda\Omega g-h\right\| \geq \left\|g\right\|$ letting λ goes to 0 we obtain $\left\|g-h\right\| \geq \left\|g\right\|$ for all $g\in D(\Omega)$ and also for all $g\in C(X)$ because $D(\Omega)$ is dense in $C(X)$; taking $g=h$ gives $\left\|h\right\|=0$

Consequently $\overline{\Gamma}$ is the graph of the closed extension $\overline{\Omega}, D\left(\overline{\Omega}\right)$ of Ω. Let us prove that $\overline{\Omega}$ is markovian. We prove only condition (*C*) of Definition 6.1, the other conditions are evident.

Let $f\in D\left(\overline{\Omega}\right), \lambda\geq 0$ and put $g=f-\lambda\overline{\Omega}f$ there exists $\left(f_n\right)\subset D(\Omega)$ such that $f_n\to f$ and $\Omega f_n\to\overline{\Omega}f$, so if $g_n=f_n-\lambda\Omega f_n$ we get $g_n\to g$; Ω being markovian, $\min_x .f_n(x)\geq .\min_x .g_n(x), \forall n$ But $f_n(x)\to f(x)$ uniformly in *x*, this implies $\min_x .f_n(x)\to \min_x .f(x)$ because f_n, f are continuous and *X* compact; likewise $.\min_x .g_n(x)\to .\min_x .g(x)$ therefore $\min_x .f(x)\geq .\min_x .g(x)$ and $\overline{\Omega}$ is markovian.

Definition

A markovian generator Ω is a closed pregenerator satisfying: $R(I-\lambda\Omega)=C(X)$ for small $\lambda>0, R(I-\lambda\Omega)$ being the range of $I-\lambda\Omega$

Proposition

(a) Any bounded pregenerator on *C*(*X*) is a markovian generator.

(b) For a markovian generator we have $R(I-\lambda\Omega)=C(X)$, for any λ>0.

Proof

(a) It is well known that any bounded operator is closed. In order to prove that $R(I-\lambda\Omega)=C(X)$, for small λ>0, we solve the equation $f-\lambda\Omega f=g$, for $g\in C(X)$ and $0<\lambda<\left\|\Omega\right\|^{-1}$. Indeed for such λ the operator $I-\lambda\Omega$ is invertible with inverse the Neumann series $\sum_{n=0}^{\infty}.\lambda^n\Omega^n$ and the equation $f-\lambda\Omega f=g$, has the solution $f=\sum_{n=0}^{\infty}.\lambda^n\Omega^n g$.

(b) First we prove the implication:

$\lambda>0, R(I-\lambda\Omega)=C(X)$ and $\lambda<\gamma\Rightarrow R(I-\gamma\Omega)=C(X)$ let $g\in C(X)$, we want to solve the equation $f-\lambda\Omega f=g$, for $f\in D(\Omega)$ let us define the linear operator $T:C(X)\to D(\Omega)$ by the recipe:

$$T(h)=\lambda\gamma^{-1}(I-\lambda\Omega)^{-1}g+(\gamma-\lambda)\gamma^{-1}(I-\lambda\Omega)^{-1}h$$

Where the inverse $(I-\lambda\Omega)^{-1}$ exists since $R(I-\lambda\Omega)=C(X)$, and Ω is dissipative. On the other hand, since Ω is dissipative we have:

$\left\|Th_1-Th_2\right\|-(\gamma-\lambda)\gamma^{-1}\left\|(I-\lambda\Omega)^{-1}(h_1-h_2)\right\|\leq(\gamma-\lambda)\gamma^{-1}\left\|(h_1-h_2)\right\|$ But $0<(\gamma-\lambda)\gamma^{-1}<1$ so T is a contraction; let *f* be its fixed point and then $Tf=f\in D(\Omega)$. Therefore $(I-\lambda\Omega)f=(I-\lambda\Omega)Tf=\lambda\gamma^{-1}g+(\gamma-\lambda)\gamma^{-1}f$ Which gives exactly $f-\lambda\Omega f=g$.

We end this section with a version of Hille-Yosida Theorem adapted to the markov semigroups context.

Theorem

There is a one to one correspondence between markovian generators Ω on *C*(*X*) and markov semigroups S_t, t≥0, on *C*(*X*). It is given by:

(1) $D(\Omega)=\left\{f\in C(X):\lim_{t\to 0}\frac{S_t f-f}{t}, exists\right\}$

(2) $f\in D(\Omega), \Omega f=\lim_{t\to 0}\frac{S_t f-f}{t}$

(3) $S_t f=\lim_{n\to\infty}\left(I-\frac{t}{n}\Omega\right)^{-n}f, f\in C(X), t\geq 0$

(4) $f\in D(\Omega), S_t f\in D(\Omega), and \frac{d}{dt}S_t f=\Omega S_t f=S_t\Omega f$

(5) If $g\in C(X), \lambda\geq 0$ then the unique solution of the equation $f-\lambda\Omega f=g$, is given by $f=\int_0^{\infty}e^{-t}S_{\lambda t}g.dt$. [5,6] for the proof.

Application

In general the operator $\Omega, D(\Omega)$ is associated to the following

Cauchy problem: $F'(t) = \Omega F(t), F(0) = f \in D(\Omega)$.

If Ω is a markovian generator and $F(t) \in D(\Omega), \forall t \geq 0$, the Hille-YosidaTheorem gives the unique solution to this problem in the form $F(t) = S_t f$ where S_t is the semigroup generated by Ω with this framework we have:

Theorem

Let Ω be the generator of a markov semigroup S_t, $t \geq 0$ on $C(X)$ and let $F(t)$, $G(t)$ be functions from [0, ∞] into $C(X)$ such that:

(i) $F(t) \in D(\Omega), \forall t \geq 0$

(ii) $G(t)$ is continuous on [0, ∞]

(iii) $F'(t) = \Omega F(t) + G(t), t, \geq 0$

then we have $F(t) = S_t F(0) + \int_0^t S_{t-s} G(s).ds$

Proof

We have $\frac{S_{t-s-h} F(s+h) - S_{t-s} F(s)}{h} = S_{t-s}\left[\frac{F(s+h) - F(s)}{h}\right] + \left[\frac{S_{t-s-h} - S_{t-s}}{h}.F(s)\right] + \left[S_{t-s-h} - S_{t-s}\right].F'(s)$

$+\left[S_{t-s-h} - S_{t-s}\right]\left[\frac{F(s+h) - F(s)}{h} - F'(s)\right]$

for $0 \leq S \leq t$ and $0 \leq h \leq t\text{-}s$ making h goes to 0 we get the left hand side goes to $\frac{d}{ds} S_{t-s} F'(s)$ for the right hand side we have the first term goes to $S_{t-s} F'(s)$ by continuity of S_{t-s} the second term goes to $-S_{t-s}\Omega F(s)$ by (i) and Theorem 6.9 the third term goes to 0 by continuity of $u \to S_u F'(s), u > 0$ the fourth term goes to 0 since S_{t-s} and S_{t-s-h} are contractions Consequently for $0 \leq S \leq t$

$\frac{d}{ds} S_{t-s} F'(s) = S_{t-s} F'(s) - S_{t-s}\Omega F(s) = S_{t-s} G(s)$ by (iii) since S_{t-s} is continuous in S and the same for $G(S)$ by (ii) we deduce the continuity of the function $s \to S_{t-s} G(s)$; we can perform the following integration $F(t) - S_t F(0) = \int_0^t S_{t-s} G(s).\text{ds}$.

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