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Exact Solutions and Conservation Laws of Dissipative Hyperbolic Mean Curvature Flow for Lagrangian Graphs

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Abstract

The classical Lie group method is used to study dissipative hyperbolic mean curvature flow for Lagrangian graphs, then based on one-dimensional optimal system of the symmetries to the equation, the reduction equations and exact solutions are calculated. Finally, we analyze the conservation laws of dissipative hyperbolic mean curvature flow for Lagrangian graphs.

Keywords: Lie Symmetry Analysis; Exact Solutions; Dissipative Hyperbolic Mean Curvature Flow for Lagrangian Graphs; Conservation laws

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1. Introduction

Hyperbolic mean curvature flow is of great significance to geometry, modern physics, crystallography and nonlinear dynamics of surface motion. In recent years, Kong and his collaborators have made a thorough study of the hyperbolic geometric flow [1-6], for example, they demonstrated some properties of the short-time existence and nonlinear stability of hyperbolic geometric flow, and gave the lower lifetime bounds for classical solutions of hyperbolic geometric flows with asymptotically flat initial Riemannian surfaces. Kong et al introduced hyperbolic mean curvature flow in [7], and proved the unique short-time smooth solution of the flow. In [8], LeFloch et al derived the hyperbolic mean curvature flow from the principle of minimum potential energy, in the meantime, they deeply studied the conditions of the flow keeping normal motion, finally obtained a new class of hyperbolic mean curvature flow and showed some properties of this flow. In [9-10], Kong et al pointed out that the hyperbolic mean curvature flow can keep convex evolution, and found the flow will blow up in finite time or produce singularity. In [11], He et al studied the self-similar solutions of the hyperbolic mean curvature flow, and further explored that all curves contained in plane which moving in a self-similar manner under the flow are straight lines or circles. In [12], by the support function of strictly closed convex curve, Ding et al transformed the one dimensional hyperbolic inverse mean curvature into hyperbolic partial differential equation, then studied the symmetries and exact solutions of the equation by the classical Lie group method.

In [13], Duan et al investigated the hyperbolic mean curvature flow for Lagrangian graphs

$$v_{tt} = \arctan v_{xx}, \tag{1}$$

then they differentiated both sides of this equation with respect to x, got the following nonlinear evolution equation

$$v_{xtt} - \frac{v_{xxx}}{1+v_{xx}^2} = 0, \tag{2}$$

under the condition of the initial data were periodic, they not only proved the C^3 solution to the above flow blows up in finite time, but also deduced its life-span.

In this paper, we will study dissipative hyperbolic mean curvature flow for Lagrangian graphs

$$v_{tt} + v_t = \arctan v_{xx}, \tag{3}$$

then we can obtain the following nonlinear evolution equation

$$v_{xtt} + v_{xt} - \frac{v_{xxx}}{1+v_{xx}^2} = 0. \tag{4}$$

The aim of this paper is to find symmetries of dissipative hyperbolic mean curvature flow for Lagrangian graphs via the classical Lie group method [14-17]. Then we investigate one-dimensional optimal system of the symmetries to the flow by using the method in [18]. After solving the reduction equations, we calculate the exact solutions of the flow. Based on the method and thought provided by Ibragimov [19], we obtain conservation laws of dissipative hyperbolic mean curvature flow for Lagrangian graphs.

In the end, this paper is arranged as follows. Section 2, we mainly discuss Lie symmetries of dissipative hyperbolic mean curvature flow for Lagrangian graphs. Section 3, the exact solutions of the flow are obtained by solving the reduction equations. Section 4, conservation laws have been set up. In the last part, conclusions are made in Section 5.

2. Lie Symmetry Analysis

In this part, we suppose

$$v_x = u, \tag{5}$$

so eq. (4) becomes

$$u_{tt} + u_t - \frac{u_{xx}}{1+u_x^2} = 0, \tag{6}$$

we use classical Lie symmetry analysis for eq. (6) and consider a transformation group with one parameter

$$\begin{aligned} x^* &= x + \varepsilon\xi(x,t,u) + o(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(x,t,u) + o(\varepsilon^2), \\ u^* &= u + \varepsilon\phi(x,t,u) + o(\varepsilon^2), \end{aligned} \tag{7}$$

in which $\xi(x,t,u), \tau(x,t,u), \phi(x,t,u)$ are all arbitrary functions of x, t, u and ε is an infinitesimal parameter. The infinitesimal vector field of eq. (6) is

$$X = \xi(x,t,u)\,\partial_x + \tau(x,t,u)\,\partial_t + \phi(x,t,u)\,\partial_u, \tag{8}$$

the form of the second order extension of infinitesimal algebra is

$$\mathrm{Pr}^{(2)} X = X + \phi^x\,\partial_x + \phi^t\,\partial_t + \phi^{xx}\,\partial_{xx} + \phi^{tt}\,\partial_{tt}, \tag{9}$$

$$\begin{aligned} \phi^x &= D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{tx}, \\ \phi^t &= D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt}, \\ \phi^{xx} &= D_{xx}(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{txx}, \\ \phi^{tt} &= D_{tt}(\phi - \xi u_x - \tau u_t) + \xi u_{xtt} + \tau u_{ttt}, \end{aligned} \tag{10}$$

from the condition of invariance, we have

$$\mathrm{Pr}^{(2)}(\triangle)\big|_{\triangle=0} = 0, \tag{11}$$

in which $\triangle = u_{tt} + u_t - \frac{u_{xx}}{1+u_x^2}$.

By solving eq. (11), we can get the overdetermined equations of eq. (6),

$$\begin{gathered} \phi_u = 0, \phi_x = 0, \phi_{tt} = -\phi_t, \\ \xi_t = \xi_u = \xi_x = 0, \\ \tau_u = \tau_x = \tau_t = 0, \end{gathered} \tag{12}$$

then solving the above equations, we have

$$\phi = c_1 + c_2 e^{-t}, \tau = c_3, \xi = c_4, \tag{13}$$

where c_1, c_2, c_3, c_4 are arbitrary constants. The corresponding infinitesimal generator of eq. (2) is

$$X = (c_1 + c_2 e^{-t})\,\partial_\phi + c_3\,\partial_t + c_4\,\partial_x. \tag{14}$$

So the infinitesimal generators of eq. (6) are spanned by

$$\begin{aligned} X_1 &= \partial_u , \\ X_2 &= e^{-t}\,\partial_u , \\ X_3 &= \partial_t , \\ X_4 &= \partial_x . \end{aligned} \tag{15}$$

The Lie transformation groups corresponding to the following generators are calculated and their initial value problems are considered

$$\begin{aligned} \frac{dx^*}{d\varepsilon} &= \xi(x^*,t^*,u^*), x^*|_{\varepsilon=0} = x , \\ \frac{dt^*}{d\varepsilon} &= \tau(x^*,t^*,u^*), t^*|_{\varepsilon=0} = t , \\ \frac{du^*}{d\varepsilon} &= \phi(x^*,t^*,u^*), u^*|_{\varepsilon=0} = u, \end{aligned} \tag{16}$$

the corresponding single parameter group are obtained by solving the above vector fields,

$$\begin{aligned} G_1 &: (x,t,u) \longrightarrow (x,t,u+\varepsilon) , \\ G_2 &: (x,t,u) \longrightarrow (x,t,u+e^{-t}\varepsilon) , \\ G_3 &: (x,t,u) \longrightarrow (x,t+\varepsilon,u) , \\ G_4 &: (x,t,u) \longrightarrow (x+\varepsilon,t,u) , \end{aligned} \tag{17}$$

so from the single parameter group $G_i(i = 1,2,3,4)$ in (17), the expression of the solution of the eq. (6) can be obtained:

$$\begin{aligned} G_1 * u(x,t) &= u(x,t) + \varepsilon , \\ G_2 * u(x,t) &= u(x,t) + e^{-t}\varepsilon , \\ G_3 * u(x,t) &= u(x,t-\varepsilon) , \\ G_4 * u(x,t) &= u(x-\varepsilon,t) . \end{aligned} \tag{18}$$

According to the definition of Lie bracket and adjoint representation,

$$[X_i, X_j] = X_iX_j - X_jX_i , \tag{19}$$

$$Ad(\exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i,X_j] + \frac{\varepsilon}{2!}[X_i,[X_i,X_j]] - \cdots , \tag{20}$$

we can get Table 1 and Table 2 respectively.

X_i, X_j	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	X_2	0
X_3	0	$-X_2$	0	0
X_4	0	0	0	0

Table 1: Commutator table

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	X_4
X_2	X_1	X_2	$X_3 - \varepsilon X_2$	X_4
X_3	X_1	$e^{\varepsilon} X_2$	X_3	X_4
X_4	X_1	X_2	X_3	X_4

Table 2: Adjoint Representation Table

3. Exact Solutions

3.1 Optimal Systems

Next, we are going to use Table 2 to give the classification of subalgebras for eq. (6), assume a vector

$$X = e_1 X_1 + e_2 X_2 + e_3 X_3 + e_4 X_4 \,. \tag{21}$$

Firstly, we suppose $e_3 \neq 0$, we select $e_3 = 1$, (21) becomes

$$X = e_1 X_1 + e_2 X_2 + X_3 + e_4 X_4 \,, \tag{22}$$

we apply X to $Ad(\exp(e_2)X_2)$, so the coefficients of X_2 can be eliminated,

$$X^{(1)} = Ad(\exp(e_2)X_2)X = e_1 X_1 + X_3 + e_4 X_4 \,, \tag{23}$$

that is to say, when $e_3 \neq 0$, X is equivalent to $X^{(1)} = X_3 + e_1 X_1 + e_4 X_4$.

The rest of one-dimensional subalgebras are generated by X with $e_3 = 0$, X is equivalent to $X^{(2)} = e_1 X_1 + e_2 X_2 + e_4 X_4$.

Based on the above discussion, we obtain the following theorem.

Theorem 3.1. Ane-dimensional subalgebras of the 4-dimensional Lie algebra is spanned by:

$$\begin{aligned}&(1).\ X_3, X_3 \pm X_1, X_3 \pm X_4,\ X_3 \pm X_1 \pm X_4, e_3 \neq 0\\&(2).\ X_1, X_2, X_4,\ X_1 \pm X_2, X_1 \pm X_4, X_2 \pm X_4, X_1 \pm X_2 \pm X_4, e_3 = 0\,.\end{aligned}$$

3.2 Symmetry reductions and invariant solutions

In this part, we mainly seek the similar reduction and invariant solutions of eq. (4).

Case 1: $X_2 = \partial_t$. By solving the characteristic equation

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0} \,, \tag{24}$$

we get the group-invariant solution

$$u = f(s) \,, \tag{25}$$

where $s = x$. Taking (25) into eq. (6), we have

$$f_{ss} = 0 \,, \tag{26}$$

by solving eq. (26), we can get

$$f = c_1 s + c_2 \,. \tag{27}$$

So

$$u = c_1 x + c_2 \,, \tag{28}$$

therefore eq. (4) has a solution

$$v = \int (c_1 x + c_2) dx = \frac{c_1 x^2}{2} + c_2 x + g(t), \tag{29}$$

where $g(t)$ is an arbitrary functions about t. The graph of (29) for

$$c_1 = 1, c_2 = 1, g(t) = \text{sech}(t) \,,$$

is shown in Figure 1.

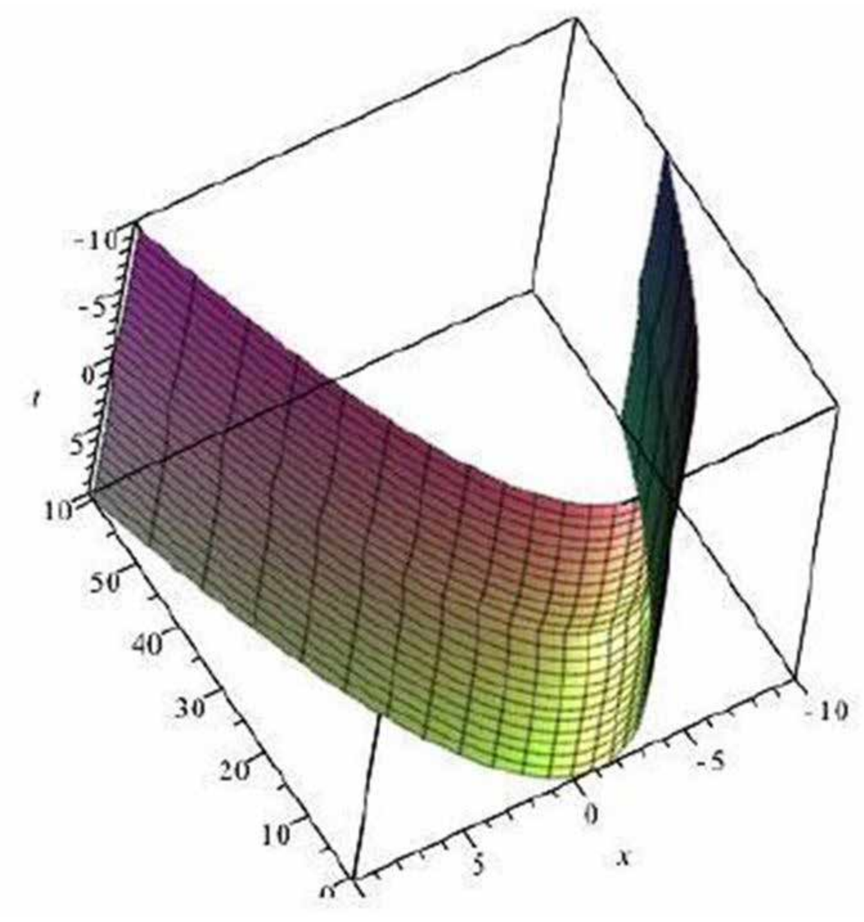

Figure 1: 3-dimensional plot

Case 2: $X_2 + X_4 = e^{-t}\partial_u + \partial_x$. Solving the characteristic equation

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{e^{-t}} , \quad (30)$$

the group-invariant solution is

$$u = e^{-s}x + f(s) , \quad (31)$$

where $s = t$. Taking (31) into eq. (6), we get

$$f_s + f_{ss} = 0 , \quad (32)$$

solving eq. (32),

$$f = c_1 e^{-s} + c_2 , \quad (33)$$

$$u = c_1 e^{-t} + c_2 + e^{-t}x , \quad (34)$$

so we can get

$$v = \int (c_1 e^{-t} + c_2 + e^{-t}x)dx = c_1 e^{-t}x + c_2 x + \frac{e^{-t}x^2}{2} + g(t), \quad (35)$$

where $g(t)$ is an arbitrary functions about t. The graph of (35) for

$$c_1 = 0, c_2 = 1, g(t) = \sin(t),$$

is shown in Figure 2.

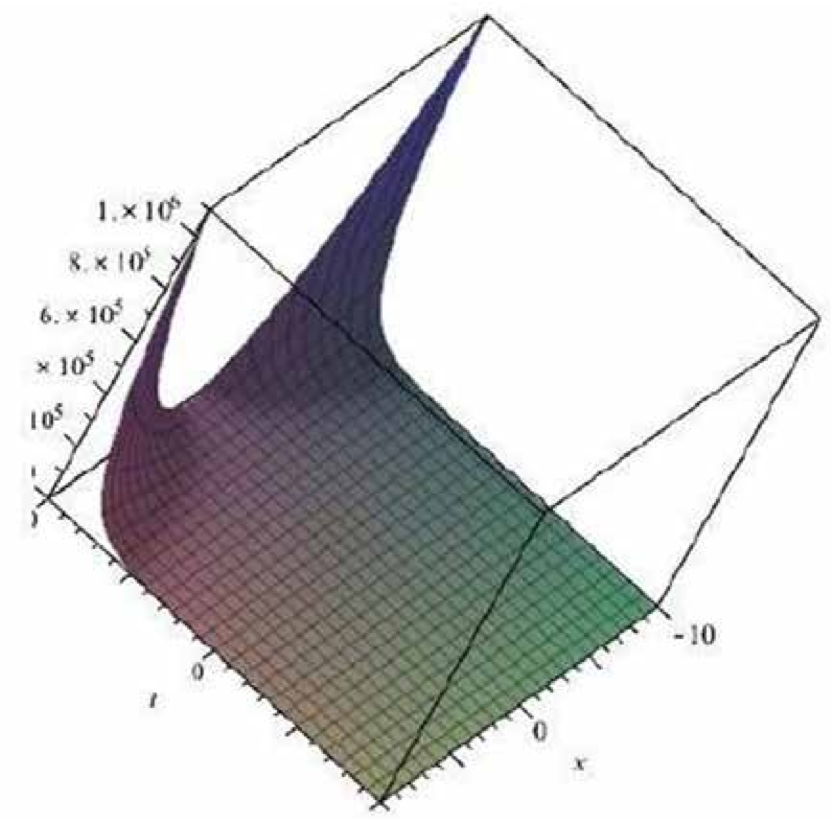

Figure 2: 3-dimensional plot

Case 3: $X_1 + X_3 + X_4 = \partial_u + \partial_t + \partial_x$. The characteristic equation is

$$\frac{dx}{1} = \frac{dt}{1} = \frac{du}{1}, \tag{36}$$

by solving eq. (36), we have

$$u = f(s) + t, \tag{37}$$

where $s = t - x$. Taking (37) into eq. (6), eq. (6) reduces into

$$1 + f_s + f_{ss} - \frac{f_{ss}}{1 + f_s^2} = 0, \tag{38}$$

then we suppose $f_s = y$, eq. (38) becomes

$$1 + y + y' - \frac{y'}{1 + y^2} = 0, \tag{39}$$

then we can find that the general solution of eq. (39) is

$$s + \frac{1}{4}\ln(y^2 + 1) - \frac{1}{2}\arctan(y) + \frac{1}{2}\ln(y + 1) + c_1 = 0, \tag{40}$$

if y is solved, we can get

$$f(s) = \int y(s)ds + c_1. \tag{41}$$

So

$$u = \int y(t - x)d(t - x) + t + c_1, \tag{42}$$

finally, we obtain solution of eq. (4) as shown

$$v = \int \left(\int y(t - x)d(t - x)\right)dx + (t + c_1)x + g(t), \tag{43}$$

where $g(t)$ is an arbitrary functions about t.

Case 4: $X_3 + X_4 = \partial_x + \partial_t$. By solving characteristic equation, we have

$$u = f(s), \tag{44}$$

where $s = t - x$. Taking (44) into eq. (6), eq. (6) reduces into

$$f_s + f_{ss} - \frac{f_{ss}}{1 + f_s^2} = 0, \tag{45}$$

then we suppose $f_s = y$, eq. (45) becomes

$$y + y' - \frac{y'}{1 + y^2} = 0, \tag{46}$$

solving it, we can get

$$y = \pm\sqrt{e^{-2s}c_1 - 1}, \tag{47}$$

so we can get

$$f(s) = \pm\int y(s)ds + c_2 = \pm\left(\arctan\left(\sqrt{e^{-2s}c_1 - 1}\right) - \sqrt{e^{-2s}c_1 - 1}\right) + c_2. \tag{48}$$

So

$$u = \pm\left(\arctan\left(\sqrt{e^{-2(t-x)}c_1 - 1}\right) - \sqrt{e^{-2(t-x)}c_1 - 1}\right) + c_2\,, \tag{49}$$

therefore, we acquire the solution of eq. (4)

$$v = \int \pm\left(\arctan\left(\sqrt{e^{-2(t-x)}c_1 - 1}\right) - \sqrt{e^{-2(t-x)}c_1 - 1}\right)dx + c_2 x + g(t)\,, \tag{50}$$

where $g(t)$ is an arbitrary functions about t. The graph of (50) for

$$c_1 = 1, c_2 = 0, g(t) = \sin(t),$$

is shown in Figure 3.

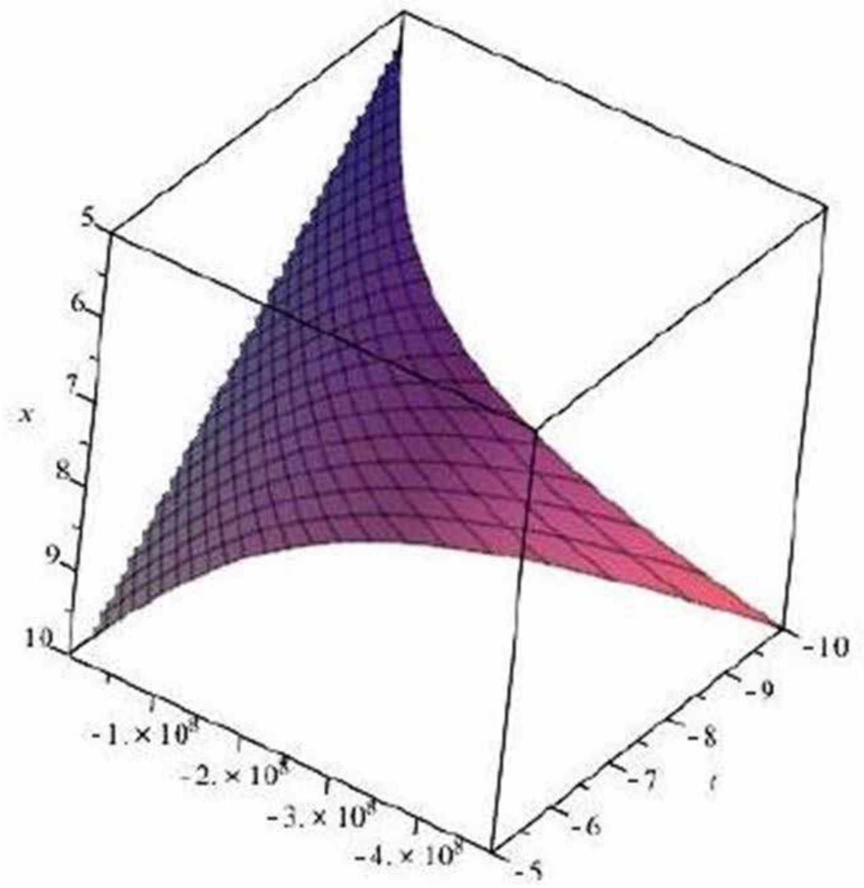

Figure 3: 3-dimensional plot

Case 5: $X_1 + X_3 = \partial_t + \partial_u$. Solving the characteristic equation, we have

$$u = f(s) + t \,, \tag{51}$$

where $s = x$. Taking (51) into eq. (6), eq. (6) becomes

$$1 - \frac{f_{ss}}{1 + f_s^2} = 0 \,, \tag{52}$$

solving (52), we get

$$f = -\ln(c_1 \sin(s) - c_2 \cos(s)) \,, \tag{53}$$

so

$$u = -\ln(c_1 \sin(x) - c_2 \cos(x)) + t \,, \tag{54}$$

therefore eq. (4) has a solution as

$$v = \int -\ln(c_1 \sin(x) - c_2 \cos(x)) dx + tx + g(t) \,, \tag{55}$$

where $g(t)$ is an arbitrary functions about t.

4. Conservation Laws

In this section, we are going to calculate the conservation laws of eq. (4). Suppose the standard Lagrange function of eq. (4) is

$$T = \eta(t,x,v)\left(v_{xt} + v_{xtt} - \frac{v_{xxx}}{1 + v_{xx}^2}\right), \tag{56}$$

where $\eta(t,x,v)$ is the conservation law multiplier, it can be given by the following determining equation

$$\frac{\delta T}{\delta v} = 0 \,, \tag{57}$$

$\frac{\delta}{\delta v}$ is the Euler-Lagrange operator, and its expression is

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_i \frac{\partial}{\partial v_i} + D_{ij} \frac{\partial}{\partial v_{ij}} - D_{ijk} \frac{\partial}{\partial v_{ijk}} + D_{ijkl} \frac{\partial}{\partial v_{ijkl}} - \cdots, \tag{58}$$

by solving eq. (57), we get

$$\frac{\partial}{\partial v}\eta(t,x,v) = 0, \frac{\partial}{\partial x}\eta(t,x,v) = 0 \,, \tag{59}$$

solving the above equations, we get

$$\eta(t,x,v) = h(t), \tag{60}$$

where $h(t)$ is an arbitrary functions about t. Here we select

$$h(t) = t\,. \tag{61}$$

So

$$T = t\left(v_{xt} + v_{xtt} - \frac{v_{xxx}}{1+v_{xx}^2}\right), \tag{62}$$

The conserved vector is given by $C = (C^x, C^t)$, where

$$\begin{aligned} C^i = \; & \xi^i T + W^\alpha \left[\frac{\partial T}{\partial v_i} - D_j\left(\frac{\partial T}{\partial v_{ij}}\right) + D_j D_k\left(\frac{\partial T}{\partial v_{ijk}}\right) - D_j D_k D_l\left(\frac{\partial T}{\partial v_{ijkl}}\right) + \cdots\right] \\ & + D_j(W^\alpha)\left[\frac{\partial T}{\partial v_{ij}} - D_k\left(\frac{\partial T}{\partial v_{ijk}}\right) + D_k D_m\left(\frac{\partial T}{\partial v_{ijkm}}\right) - \cdots\right] \\ & + D_j D_k(W^\alpha)\left[\frac{\partial T}{\partial v_{ijk}} - D_m\left(\frac{\partial T}{\partial v_{ijkm}}\right) + \cdots\right] \\ & + D_j D_k D_m(W^\alpha)\left[\frac{\partial T}{\partial v_{ijkm}} - D_n\left(\frac{\partial T}{\partial v_{ijkmn}}\right) + \cdots\right], \end{aligned} \tag{63}$$

where $W^\alpha = \eta^\alpha - \xi^j v_j^\alpha$. The conservation law for eq. (4) is determined by the following formula:

$$D_x(C^x) + D_t(C^t) = 0, \tag{64}$$

where

$$\begin{aligned} C^x = \; & \xi^x T + W\left[-D_x\left(\frac{\partial T}{\partial v_{xx}}\right) - D_t\left(\frac{\partial T}{\partial v_{xt}}\right) + D_x D_x\left(\frac{\partial T}{\partial v_{xxx}}\right) + D_t D_t\left(\frac{\partial T}{\partial v_{xtt}}\right)\right] + D_t(W)\left[\frac{\partial T}{\partial v_{xt}} - D_t\left(\frac{\partial T}{\partial v_{xtt}}\right)\right] \\ & + D_x(W)\left[\frac{\partial T}{\partial v_{xx}} - D_x\left(\frac{\partial T}{\partial v_{xxx}}\right)\right] + D_x D_x(W)\left(\frac{\partial T}{\partial v_{xxx}}\right) + D_t D_t(W)\left(\frac{\partial T}{\partial v_{xtt}}\right), \end{aligned} \tag{65}$$

$$\begin{aligned} C^t = \; & \xi^t T + W\left[-D_x\left(\frac{\partial T}{\partial v_{xt}}\right) + D_x D_t\left(\frac{\partial T}{\partial v_{xtt}}\right)\right] + D_x(W)\left[\left(\frac{\partial T}{\partial v_{xt}}\right) - D_t\left(\frac{\partial T}{\partial v_{xtt}}\right)\right] + D_t(W)\left[-D_x\left(\frac{\partial T}{\partial v_{xtt}}\right)\right] \\ & + D_x D_t(W)\left[\frac{\partial T}{\partial v_{xtt}}\right]. \end{aligned} \tag{66}$$

Let's take the vector field X_3 as an example:

$$C^x = v_t + v_{tt}(t-1) + v_{tt} + t v_{ttt}\,, \tag{67}$$
$$C^t = -v_{tx}(t-1) - t v_{ttx}\,. \tag{68}$$

5. Conclusion

In this paper, the Lie symmetry analysis is performed on dissipative hyperbolic mean curvature flow for Lagrangian graphs, then by solving the reduction equations, the exact solutions of the flow are carried out. Eventually, the conservation laws of dissipative hyperbolic mean curvature flow for Lagrangian graphs are solved.

Remark 1: Compared with [11], He et al concerned the self-similar solutions to the hyperbolic mean curvature flow for plane curves. They proved that all curves immersed in the plane which move in a self-similar manner under the hyperbolic mean curvature flow are straight lines and circles. Compared with [12] and [17], we studied different equations by using Lie group method, and the ways to find the optimal system were different. Compared with [13], Duan et al assumed the initial data were periodic, then they found the C^3 solution to the hyperbolic mean curvature flow for Lagrangian graphs must blow up in finite time and the lifespan can be also derived.

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