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A Note on Selection between Three Choices

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Abstract

Suppose that a sample space with three objects is given. In this case, the problem of selecting the most probable objects by voting system is studied. It is assumed that there is a type of inter-correlation between two objects. Indeed, one of them may be selected by voters to make sure that the other will not be selected. First, it is seen that this inter-correlation causes a spurious interpretation of observed probability of selection for each object. Next, some game theoretic settings between objects are proposed and the Nash equilibriums are derived. Interestingly, it is seen that the Nash equilibrium probabilities are equal to the hidden actual probabilities. Learning schemes are proposed. Bayesian equilibriums are derived. Finally, a conclusion section is also given.

Keywords: Actual and Observed Probability; Bayesian Equilibrium; Game Theory; Hidden Probability; Inter-Correlation; Nash Equilibrium

Introduction

The paper deals with the problem of selecting the most probable objects (candidates) by a voting system with three candidates, in which the main competition is between two of them and objects of sample space are inter-correlated. That is, one of candidates is selected by voters to make sure that the other will not be selected. Therefore, naturally, both main candidates do their best tries to motivate people to vote to them and advocators of each candidate motivate voters to vote the specific candidate to make sure that the other main candidate is not selected. This fact changes the voting environment to a strategic environment which needs the game theory tools for analyzing it. An interesting consequence of this situation is that the observed probability of selection for each object is larger than the actual probability, spuriously. For a motivating real-life example, some presidential elections of some countries in Middle East region are suspected that a same phenomenon has happened. Interested readers may refer to related historical reports frequently appeared in web. It is seen that the derived Nash equilibrium probabilities are equal to the hidden actual probabilities. To obtain the Nash equilibriums the learning schemes and Bayesian equilibriums are given. To this end, consider the problem of choosing between three objects (candidates) *a, b, c* of sample space $S = \{a,b,c\}$ using a voting system. The hidden actual probability of selection of i-th candidate is $p_i \in [0,1]$, $i=a,b,c$ and $\sum_{i=a,b,c} p_i = 1$. Suppose that the main competition is between candidates *a* and *b* and it is assumed that there is a inter-correlation between candidates *a, b* in this sense. Indeed, the advocators of candidate *a (b)* encourage people to vote candidate *a (b)* for two reasons, first *a (b)* is selected and second to make sure that *b (a)* is not selected. Thus, the observed (not actual) probability of selecting candidate *a* is spurious number $p_{a|\sim b} = \frac{p_a}{1-p_b} > p_a$ which is greater than the hidden actual probability p_a. Notation $p_{a|\sim b}$ stands for the probability of selecting *a* while *b* is not selected (Bernardo and Smith, 1994). Similarly, p_a is replaced with $p_{b|\sim a} = \frac{p_b}{1-p_a} > p_b$. The following two facts are interesting [1].

Two facts

1) Notice that, $p_{a|\sim b} - p_a = \frac{p_b}{1-p_b} p_a$. As $p_b \to 0$, this difference becomes negligible and as $p_b \to 1$, this difference gets larger.

2) Notice that $p_{a|\sim b} - p_{b|\sim a} > 0$ (< 0) if and only if $p_a - p_b > 0$ (< 0). That is, the observed winner is the actual winner but his/her probability is bigger), spuriously, which has fundamental effects on decisions of winner since it causes big mistakes for him/her about the percentage of his/her votes.

Game theoretic setting

Next, consider a game between two candidates a,b with utilities

$$\begin{cases} u_a(p_a, p_b) = p_{a|\sim b} = \dfrac{p_a}{1-p_b}, \\ u_b(p_b, p_a) = p_{b|\sim a} = \dfrac{p_b}{1-p_a}, \end{cases} \tag{1}$$

$0 \le p_a + p_b \le 1$ at which candidate *a (b)* decides to maximize $u_a(p_a, p_b)$ $(u_b(p_b, p_a))$ with respect to p_a (p_b). It is easy to see that the best response function of formula (1) (Osborne and Rubinstein, 1994) of candidate *a (b)* when candidate *b (a)* does his/her best action p_b^* (p_a^*) is $1-p_b^*$ $(1-p_a^*)$. In this case $p_c^* = 0$ and $p_{a|\sim b}$ and $p_{b|\sim a}$ receive to their maximum value which are one. In the whole of paper, notations $p_i^*, i = a,b,c$ stand for the Nash equilibrium probabilities, see Gibbons (1992) [2].

However, the above result may be corrected, since it should be assumed that $p_{a|\sim b} + p_{b|\sim a} \le 1$, since the observed probability of selection of candidate c is $\varepsilon = 1 - p_{a|\sim b} - p_{b|\sim a}$. It is also easy to see that $p_{a|\sim b} + p_{b|\sim a} \le 1$ if and only if $p_a p_b \le p_c^2$. Again, the best response function of candidate a (b) is $\frac{p_c^2}{p_b^*}$ $(\frac{p_c^2}{p_a^*})$. Thus, the Nash equilibrium occurs at $p_a^* p_b^* = p_c^2$. Here, $p_{a|\sim b} + p_{b|\sim a} = 1$ and the observed (not actual) probability of candidate c is zero. Thus, candidate a (b) success the competition if and only if $p_{a|\sim b} > 0.5$ $(p_{b|\sim a} > 0.5)$ or equivalently $(p_{b|\sim a} > 0.5)$ $(p_b^* > p_c)$. To see why, notice that $p_{a|\sim b} = \frac{p_a}{1-p_b} = \frac{p_a}{p_a + p_c} > 0.5$ concludes that $p_a > p_c$. However, since p_c is kept fixed, the equilibrium probability p_a^* should be greater than p_c. Thus, in this case, $p_a^* > p_c > p_b^*$ $(p_b^* > p_c > p_a^*)$. the following proposition summarizes the above results. It is worth mentioning that all propositions exist in the current paper are novelty provided by the author. Thus, there is no need for providing associated reference.

Proposition 1: In a two candidates game with utilities $u_a(p_a, p_b)$ and $u_b(p_b, p_a)$, then 1-2 are true.
1) The Nash equilibrium occurs at

$$p_a^* p_b^* = p_c^2. \tag{2}$$

Here, $p_{a|\sim b} + p_{b|\sim a} = 1 (or\, \varepsilon = 0)$ and the observed (not actual) probability of candidate c is zero.
2) Candidate a (b) success the competition if and only if $p_{a|\sim b} > 0.5$ $(p_{b|\sim a} > 0.5)$ or equivalently $p_a^* > p_c (p_b^* > p_c)$. Then, $p_a^* > p_c > p_b^* (p_b^* > p_c > p_a^*)$.

Proof: For part 1, notice that the best response function of candidate a (b) is $\frac{p_c^2}{p_b^*}$ $(\frac{p_c^2}{p_a^*})$. Thus, the Nash equilibrium occurs at formula (2) $p_a^* p_b^* = p_c^2$. For part 2, notice that candidate a (b) success the competition if and only if $p_{a|\sim b} > 0.5$ $(p_{b|\sim a} > 0.5)$ or equivalently $p_a^* > p_c$ $(p_b^* > p_c)$.

The rest of paper is organized as follows. In the next section, other types of Nash equilibriums are derived and the learning schemes are studied. Also sequential equilibriums are considered. The Bayesian Nash type of equilibrium is studied in section 3. Finally, conclusions are given in section 4.

Equilibriums and learning schemes

In this section, other Nash equilibriums are derived and learning schemes are proposed. The sequential equilibriums are also studied.

Other equilibriums

The Nash equilibrium of previous section occurs if $p_{a|\sim b} + p_{b|\sim a} = 1$. However, in practice, the observed probability for candidate c is not zero and it is a small positive number ε. Assuming $p_c > \varepsilon$, then $p_{a|\sim b} + p_{b|\sim a} = 1 - \varepsilon$. Again, one can see that the Nash equilibrium occurs at

$$\frac{p_c^2 - p_a^* p_b^*}{p_c + p_a^* p_b^*} = \varepsilon$$

This is equivalent to $p_a^* p_b^* = \frac{p_c(p_c - \varepsilon)}{(1+\varepsilon)}$. Here, candidate a (b) success if and only if

$p_{a|\sim b} > \frac{1-\varepsilon}{2}$ $(p_{b|\sim a} > \frac{1-\varepsilon}{2})$. Thus, $p_a^* > \frac{1-\varepsilon}{1+\varepsilon} p_c$ $(p_b^* > \frac{1-\varepsilon}{1+\varepsilon} p_c)$.

Hereafter, the hidden actual probabilities and equilibrium probabilities are computed based on observed probabilities. To this end, let $p_{a|\sim b}=\pi_a$ and $p_{b|\sim a}=\pi_b$. It is easy to see that $p_a=\pi_a(1-\pi_b)/(1-\pi_a\pi_b)$, $p_b=\pi_b(1-\pi_a)/(1-\pi_a\pi_b)$ and $p_c=(\varepsilon+\pi_a\pi_b)/(1-\pi_a\pi_b)$. Then,

$$\begin{cases} p_a^*+p_b^*=\dfrac{1-\varepsilon-2\pi_a\pi_b}{\pi_a\pi_b} \\ p_a^*p_b^* \quad \dfrac{(\varepsilon \quad \pi_a\pi_b)}{(1 \quad \pi_a\pi_b)}\pi_a\pi_b \end{cases} \tag{3}$$

Thus, $p_a^*,p_b^*=0.5\left(p_a^*+p_b^*\right)\pm\sqrt{\dfrac{\left(p_a^*+p_b^*\right)^2}{4}-p_a^*p_b^*}$.

Example 1: A similar case happened in a presidential election of one of countries of the Middle East region. There, $\pi_a=0.5714$, $\pi_b=0.3828$ and $\varepsilon=0.0458$. Hence, $p_a=0.4514$, $p_b=0.21$ and $p_c=0.3386$. It is seen that $p_a^*=0.4515$, $p_b^*=0.21$. Interestingly, the Nash equilibrium probabilities are exactly equal to hidden actual probabilities. The following proposition summarizes the above discussion.

Proposition 2: Assuming $p_{a|\sim b}=\pi_a$, $p_{b|\sim a}=\pi_b$ and $\varepsilon=1-\pi_a-\pi_b$, then 1-3 are true.

1) $p_a=\pi_a(1-\pi_b)/(1-\pi_a\pi_b)$, $p_b=\pi_b(1-\pi_a)/(1-\pi_a\pi_b)$ and $p_c=(\varepsilon+\pi_a\pi_b)/(1-\pi_a\pi_b)$
2) The Nash equilibriums occur at p_a^* and p_b^* such that, as formula (3), then

$$\begin{cases} p_a^*+p_b^*=\dfrac{1-\varepsilon-2\pi_a\pi_b}{1-\pi_a\pi_b}, \\ p_a^*p_b^*=\dfrac{(\varepsilon+\pi_a\pi_b)}{(1-\pi_a\pi_b)^2}\pi_a\pi_b. \end{cases}$$

3) $p_a^*,p_b^*=0.5\left(p_a^*+p_b^*\right)\pm\sqrt{\dfrac{\left(p_a^*+p_b^*\right)^2}{4}-p_a^*p_b^*}$.

Proof: Using $p_a=\pi_a(1-\pi_b)/(1-\pi_a\pi_b)$, $p_b=\pi_b(1-\pi_a)/(1-\pi_a\pi_b)$ and $p_c=(\varepsilon+\pi_a\pi_b)/(1-\pi_a\pi_b)$, the proof of all parts are straightforward.

The next proposition states that $p_i^*=p_i, i=a,b$. To this end, notice that $p_a+p_b=(\pi_a+\pi_b-2\pi_a\pi_b)/(1-\pi_a\pi_b)=p_a^*+p_b^*$. Also, one can see that $p_ap_b=p_a^*p_b^*$.

Proposition 3: Nash equilibriums and hidden actual probabilities are equal, that is $p_i^*=p_i, i=a,b$.

Proof: Since $p_a+p_b=(\pi_a+\pi_b-2\pi_a\pi_b)/(1-\pi_a\pi_b)=p_a^*+p_b^*$ and $p_ap_b=p_a^*p_b^*$, then $p_i^*=p_i, i=a,b$. Here, the closeness of π_i to $p_i, i=a,b$ is studied. To this end, consider differences $e_i=\pi_i-p_i, i=a,b$. Notice that $e_a=\frac{p_b}{1-p_b}p_a$ and $e_b=\frac{p_a}{1-p_a}p_b$ are independent of p_b and p_a, respectively, if and only if $p_a=p_b=0.5$. Minimizing e_a and e_b with respect to p_b and p_a, respectively, defines a game theory problem. Also, notice that $p_{a|\sim b}+p_{b|\sim a}=1-\varepsilon$. Thus, the new utilities are given by

$$\begin{cases} u_a'\left(p_b,p_a\right)=e_a, \\ u_b'\left(p_a,p_b\right)=e_b, \end{cases}$$

such that $p_{a|\sim b}+p_{b|\sim a}=1-\varepsilon$. Notice that $e_a=(1-\varepsilon)p_b-\dfrac{p_b^2}{1-p_a}$ which takes its minimum which is zero at $p_b=0$ and $p_b=(1-\varepsilon)(1-p_a)$. As well as, e_b is minimum at $p_a=0$ and $p_a=(1-\varepsilon)(1-p_b)$. Thus, the Nash equilibriums occur at $p_a^*=p_b^*=0$ and

$$\begin{cases} p_b+(1-\varepsilon)p_a=1-\varepsilon, \\ p_a+(1-\varepsilon)p_b=1-\varepsilon. \end{cases} \tag{4}$$

It is seen that $p_a^* = p_b^* = \frac{1-\varepsilon}{2-\varepsilon}$. The following proposition summarizes this result.

Proposition 4: The two person game by minimizing $u_a^{'}\left(p_b, p_a\right)$ and $u_b^{'}\left(p_a, p_b\right)$ with respect to p_b, p_a, respectively, has two Nash equilibriums $p_a^* = p_b^* = 0$ and $p_a^* = p_b^* = \frac{1-\varepsilon}{2-\varepsilon}$.

Proof: Using formula (4), notice that the Nash equilibriums occur at $p_a^* = p_b^* = 0$ and

$$\begin{cases} p_b + (1-\varepsilon) p_a = 1-\varepsilon, \\ p_a + (1-\varepsilon) p_b = 1-\varepsilon. \end{cases}$$

Thus, $p_a^* = p_b^* = \frac{1-\varepsilon}{2-\varepsilon}$.

Here after, the game theory setting is studied based on the observed probabilities. To this end, notice that $e_a = \frac{\pi_a(1-\pi_a)}{1-\pi_a\pi_b}\pi_b$ and $e_b = \frac{\pi_b(1-\pi_b)}{1-\pi_a\pi_b}\pi_a$. These differences for example 1 are $e_a = 0.12$ and $e_b = 0.1728$. The new utilities are

$$\begin{cases} u_a^{"}\left(\pi_a, \pi_b\right) = \dfrac{\pi_a(1-\pi_a)\pi_b}{1-\pi_a\pi_b}, \\ u_b^{"}\left(\pi_b, \pi_a\right) = \dfrac{\pi_b(1-\pi_b)\pi_a}{1-\pi_a\pi_b}. \end{cases}$$

Notice that $\frac{\partial u_a^{"}}{\partial \pi_a} = 1 - 2\pi_a + \pi_a^2\pi_b = 0$ implies that $\pi_a^2 - \frac{2}{\pi_b}\pi_a + \frac{1}{\pi_b^2} + 1 - \frac{1}{\pi_b^2} = 0$. Thus, $(\pi_a - \frac{1}{\pi_b})^2 = \frac{1}{\pi_b^2} - 1$. It is seen that $\pi_a = \frac{1}{\pi_b} - \frac{\sqrt{1-\pi_b^2}}{\pi_b} = \frac{1}{1+\sqrt{1-\pi_b^2}}$.

Thus, the best response of candidate *a (b)* when the candidate *b (a)* does his best action π_b^* (π_a^*) is $\pi_a^* = \frac{1}{1+\sqrt{1-\pi_b^{*2}}}$ $\pi_b^* = \frac{1}{1+\sqrt{1-\pi_a^{*2}}}$. Again, the results are utilized in the following proposition.

Proposition 5: Based on observed probabilities π_a, π_b, the two person game with utilities $u_a^{"}\left(\pi_a, \pi_b\right)$ and $u_b^{"}\left(\pi_b, \pi_a\right)$, the Nash equilibriums are

$$\begin{cases} \pi_a^* = \dfrac{1}{1+\sqrt{1-\pi_b^{*2}}}, \\ \pi_b^* = \dfrac{1}{1+\sqrt{1-\pi_a^{*2}}}. \end{cases} \tag{5}$$

Using formula (5), letting $\pi_b^* = 0.3828$, it is seen that $\pi_b^* = 0.5198$. That is, the observed probabilities π_a, π_b are too close to their equilibrium values.

Learning schemes

The learning schemes appear when a specified game is repeated (Fudenberg and Tirol, 1991) [3]. To compute (π_a^*), π_b^*, four algorithms are proposed as follows.

a) Partial best response (pbr) algorithm: The pbr algorithm (Fudenberg and Tirol, 1991) for candidate *a (b)* at stage *i*-th, is the best response of current candidate to best response of player *b (a)* at stage $i-1$-th. Therefore, the following formula (6) is derived:

$$\begin{cases} \pi^*_{a,(i+1)} = (1-\lambda)\pi^*_{a,i} + \dfrac{\lambda}{1+\sqrt{1-\pi^*_{b,i}}}, \\ \pi^*_{b,(i+1)} = (1-\lambda)\pi^*_{b,i} + \dfrac{\lambda}{1+\sqrt{1-\pi^*_{a,i}}}. \end{cases} \tag{6}$$

for some forgetting factor λ. Table 1 gives the values of π^*_a for various selections of λ [3].

λ	**0.01**	**0.03**	**0.05**	**0.08**	**0.1**
π^*_a	0.355	0.507	0.536	0.543	0.543

Table 1: Values of PBR π^*_a for various λ

b) Gradient ascent algorithm: Here, following Singh *et al.* (2013), the gradient ascent learning is applied when above the two people game is played, iteratively [4]. First, notice that

$$\begin{cases} \dfrac{\partial e_a}{\partial \pi_a} = \dfrac{\pi_b(1-2\pi_a+\pi_a^2\pi_b)}{(1-\pi_a\pi_b)^2}, \\ \dfrac{\partial e_b}{\partial \pi_b} = \dfrac{\pi_a(1-2\pi_b+\pi_b^2\pi_a)}{(1-\pi_a\pi_b)^2}. \end{cases}$$

Thus, for some step size δ, iterative probabilities are given by following formula (7),

$$\begin{cases} \pi^*_{a,(i+1)} = \pi^*_{a,i} + \delta \dfrac{\partial e_a}{\partial \pi_a}\Big|_{\pi^*_{a,i},\pi^*_{b,i}}, \\ \pi^*_{b,(i+1)} = \pi^*_{b,i} + \delta \dfrac{\partial e_b}{\partial \pi_b}\Big|_{\pi^*_{b,i},\pi^*_{a,i}}. \end{cases} \tag{7}$$

As follows, time series plot of $\pi^*_{a,i}$ is given by step size $\delta = 0.1$, Figure 1. It is seen that, iterated algorithm is faster than Singh *et al.* (2013) method [4]. The y-axis is $\pi^*_{a,i}$ and x-axis is the number of observations, *i=1,…,10*. Here, converged π^*_a for various choices of δ are given in the following Table.

δ	**0.01**	**0.04**	**0.05**	**0.1**	**0.5**
π^*_a	0.271	0.576	0.606	0.618	0.618

Table 2: Values of gradient ascent π^*_a for various δ

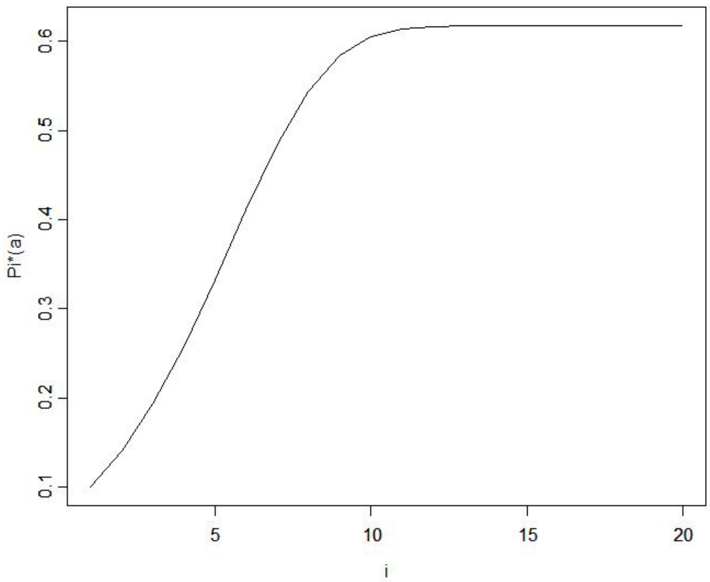

Figure 1: Equilibrium probability $\pi^*_{a,i}$: Gradient ascent learning schemes

c) Newton-Raphson (NR) algorithm: To propose the NR learning notice that, using the following formula (8), we have

$$\left(\frac{1-\pi_a^*}{\pi_a^*}\right)^2 = 1-\pi_b^* = 1-\frac{1}{1+\sqrt{1-\pi_a^*}} = \frac{\sqrt{1-\pi_a^*}}{1+\sqrt{1-\pi_a^*}} \tag{8}$$

Let $x = 1-\pi_a^*$. Then, $\left(\frac{x}{1-x}\right)^2 = \frac{\sqrt{x}}{1+\sqrt{x}}$. The last equation is equivalent to $x^{3/2}+2x-1=0$. The NR method implies that the following formula (9) is given by

$$x_{i+1} = x_i - \frac{x_i^{\frac{3}{2}}+2x_i-1}{\frac{3}{2}x_i^{\frac{1}{2}}+2}, \tag{9}$$

Where $\pi_{a,i}^* = 1-x_i$. The following plot (Figure 2) gives the NR sequence with initial value $\pi_{a,i}^* = 0.1$ which converges to 0.618. The y-axis is $\pi_{a,i}^*$ and x-axis is the number of observations, *i=1,…,10.*

d) Iterative algorithm: Here, an iterated algorithm, based on best response functions of two candidates is proposed as follows given by formula (10)

$$\begin{cases} \pi_{a,(i+1)}^* = \dfrac{1}{1+\sqrt{1-\pi_{b,i}^*}}, \\ \pi_{b,(i+1)}^* = \dfrac{1}{1+\sqrt{1-\pi_{a,i}^*}}. \end{cases} \tag{10}$$

As follows, with initial values of $\pi_{a,1}^* = \pi_{b,1}^* = 0.1$, the time series plot of $\pi_{a,i}^*, i=1,\ldots,10$ is plotted in Figure 3. It is seen that its convergence is too fast to 0.5439. The y-axis is $\pi_{a,i}^*$ and x-axis is the number of observations, $i=1,\ldots,10$.

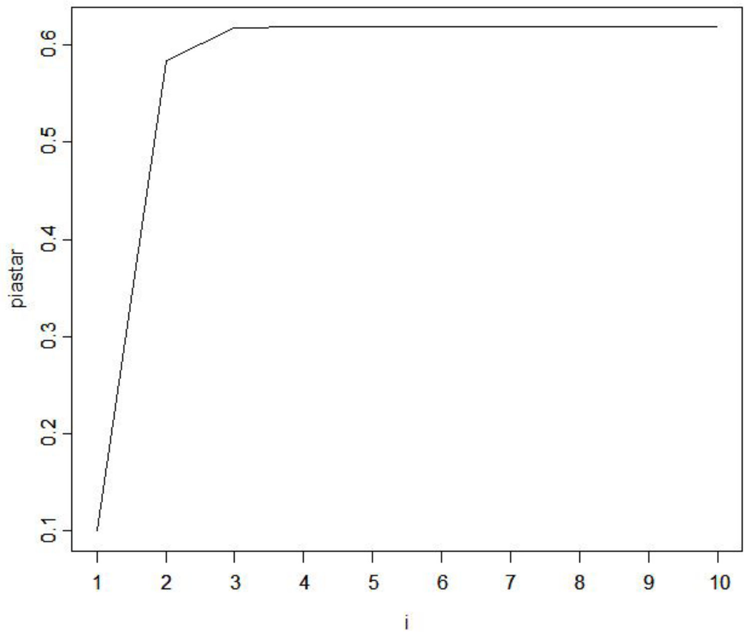

Figure 2: Equilibrium probability $\pi_{a,i}^*$: NR learning schemes

The following Table (Table 3) gives the comparison of *(a)-(d)* learning methods. Convergent is the limiting point that the specified method converges and speed is the number of iterations that method converges. The fixed tolerance for all convergences if ± 0.0001. It is seen that methods *(a)* and *(d)* are close to actual probability 0.5714 while *(b)* and *(c)* are farther than *(a), (d)*. Based on speed, between *(a)* and *(d)*, method *(d)* is selected and *(c)* is better than *(b)*. Thus, the performances of methods are sorted as $(b)<(c)<(d)<(a)$. Notice that, here, the methods *(b)* and *(c)* do not work well so it doesn't make sense to compare methods by looking at the speeds of convergence. However, it is worth mentioning that, it is only a comparison between methods for monitoring the speed of convergence to their limiting points (although, these limiting points are different for methods). Also, about choosing initial points in learning schemes, it should be stated that these results are derived for many selective initial points and almost all results were the same.

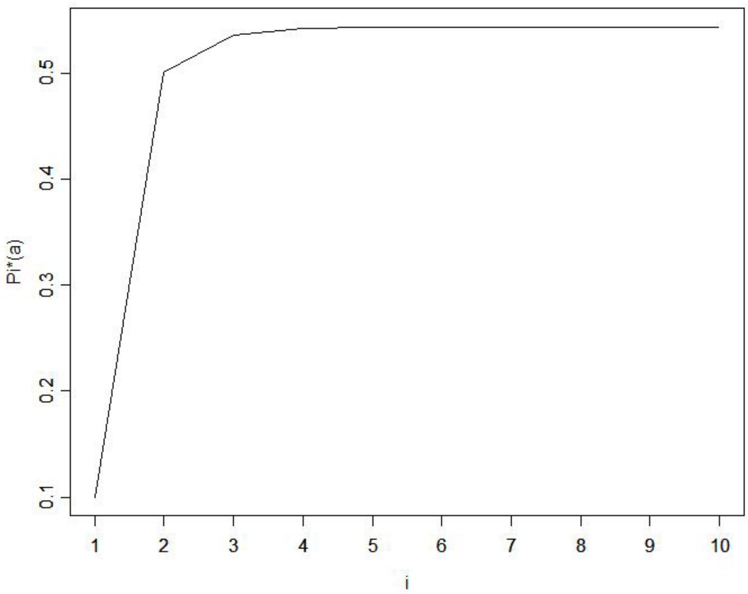

Figure 3: Equilibrium probability $\pi^*_{a,i}$: Iterated learning schemes

Proposition 6: The learning methods are given in the following Table.

Method	*(a)*	*(b)*	*(c)*	*(d)*
Convergent	0.544	0.618	0.618	0.544
Speed	14	20	4	10

Table 3: Comparison of learning schemes

Method	Recursive relation	Order of convergence
(a)	$\begin{cases} \pi^*_{a,(i+1)} = (1-\lambda)\pi^*_{a,i} + \dfrac{\lambda}{1+\sqrt{1-\pi^*_{b,i}}}, \\ \pi^*_{b,(i+1)} = (1-\lambda)\pi^*_{b,i} + \dfrac{\lambda}{1+\sqrt{1-\pi^*_{a,i}}}. \end{cases}$	$O(\lambda^{-n})$
(b)	$\begin{cases} \pi^*_{a,(i+1)} = \pi^*_{a,i} + \delta \dfrac{\partial e_a}{\partial \pi_a}\Big\vert_{\pi^*_{a,i},\pi^*_{b,i}}, \\ \pi^*_{b,(i+1)} = \pi^*_{b,i} + \delta \dfrac{\partial e_b}{\partial \pi_b}\Big\vert_{\pi^*_{b,i},\pi^*_{a,i}}. \end{cases}$	$O(\delta^{-n})$
(c)	$x_{i+1} = x_i - \dfrac{x_i^{\frac{3}{2}} + 2x_i - 1}{\frac{3}{2}x_i^{\frac{1}{2}} + 2},$	$O(?^{(-n)})$, some $\theta > 0$
(d)	$\begin{cases} \pi^*_{a,(i+1)} = \dfrac{1}{1+\sqrt{1-\pi^*_{b,i}}}, \\ \pi^*_{b,(i+1)} = \dfrac{1}{1+\sqrt{1-\pi^*_{a,i}}}. \end{cases}$	$O(1)$

Table 4: Learning methods

Sequential equilibrium

The above results are derived using the simultaneous competition of both candidates. However, this assumption may be violated, in practice, and two candidates play, sequentially, see Gibbons (1992) [2]. Suppose that, for example, candidate a is leader and candidate b is follower. Two cases are studied. First, each candidate wants to maximize the actual probability of his/her selecting. In the second case, each candidate is interested to minimize the distance between observed and actual probabilities.

(a) Maximum probability: To maximize the actual probability $p_a = \dfrac{\pi_a(1-\pi_b)}{1-\pi_a\pi_b}$ and $p_b = \dfrac{\pi_b(1-\pi_a)}{1-\pi_a\pi_b}$, sequentially, notice that $\dfrac{\partial p_b}{\partial \pi_b} = \dfrac{1-\pi_a}{(1-\pi_a\pi_b)^2} > 0$. Thus, it is enough to let $\pi_b = 1-\varepsilon-\pi_a$ which is its maximum.

As well as, $\dfrac{\partial p_a}{\partial \pi_a} = \dfrac{(1+\varepsilon)-(1-\pi_a)^2}{(1+\pi_a(1-\varepsilon-\pi_a))^2} > 0$. Thus, it is enough, to let $\pi_a = 1-\varepsilon$ and $\pi_b = 0$.

By this selection, $p_a = 1-\varepsilon$, $p_b = 0$ and $p_c = \varepsilon$

(b) Minimum distance: Here, each candidate is interested to minimize distance $e_i, i = a, b$. Following previous section, the best response of player *b* is $\pi_b^* = \dfrac{1}{1+\sqrt{1-\pi_a^{*2}}}$

Here, $\dfrac{\partial e_a}{\partial \pi_a^*} = 0$ implies that

$$\left\{\left(1-2\pi_a^*\right)\pi_b^* + \frac{\partial \pi_b^*}{\partial \pi_a^*}\pi_a^*\left(1-\pi_a^*\right)\right\}\left\{1-\pi_a^*\pi_b^*\right\} + \{\pi_b^* + \pi_a^*\frac{\partial \pi_b^*}{\partial \pi_a^*}\}\pi_a^*\left(1-\pi_a^*\right)\pi_b^* = 0.$$

Simplifying the above equation, it is concluded that formula (11) is given by

$$\pi_a^{*2} + \frac{\pi_a^*}{\sqrt{1-\pi_a^{*2}}} + \left(1-2\pi_a^*\right)\left(1+\sqrt{1-\pi_a^{*2}}\right) = 0 \tag{11}$$

Clearly, $\pi_a^* \to 1^-$ is an equilibrium the other roots satisfy $(1-\pi_a^*)\sqrt{1-\pi_a^{*2}} - \pi_a^*(1-\pi_a^*) + +1 = 0$. Then, the Newton-Raphson method with initial value 0.7 is applied. The equilibrium value of π_a^* is 0.8393 and the following plot (Figure 4) is given. Supposing $\varepsilon = 0.04$, then $\pi_b^* = 0.1207$. The y-axis is $\pi_{a,i}^*$ and x-axis is the number of observations, $i = 1, \ldots, 100$.

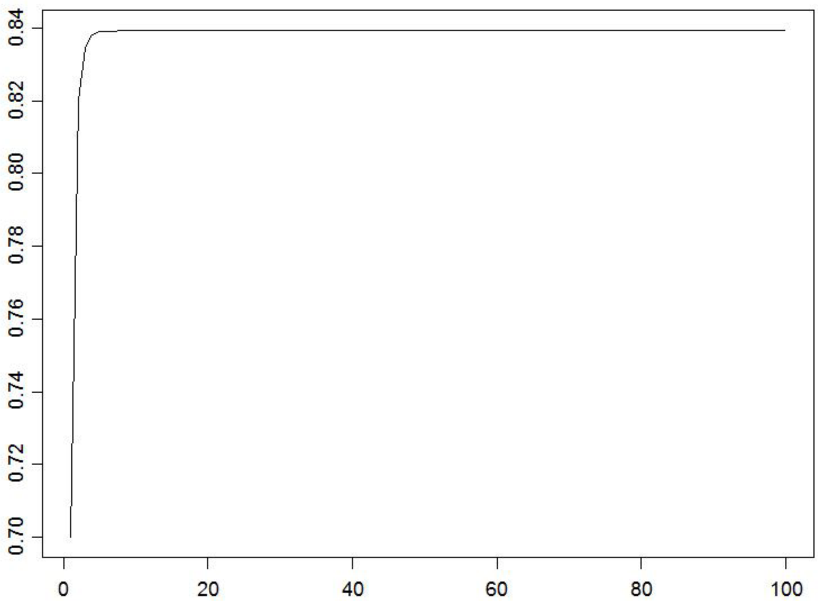

Figure 4: Equilibrium probability $\pi_{a,i}^*$: Sequential learning schemes

Bayesian equilibrium

The Bayesian game are appeared in games with incomplete information [3]. In previous sections, it is assumed that is known value. However, it is not true, necessarily, in practice. Here, it is assumed ε is a random variables with specified distribution and related equilibriums are studied. To this end, suppose that ε has a beta distribution with parameters α, β. Then $\pi_a^* + \pi_b^* = \Delta = 1-\varepsilon$ has beta distribution with parameters β, α.

Also, notice that in a simultaneous game, we have

$$\pi_a^* = \frac{1}{1+\sqrt{1-\pi_b^{*2}}}$$

Then, one can see that

$$f\left(\pi_a^*\right) = \pi_a^{*4} + 2\pi_a^{*3}\Delta - \Delta^2\pi_a^{*2} - 2\pi_a^* + 1 = 0.$$

The Newton-Raphson method implies that formula (12) is given by

$$\pi_{a,(i+1)}^* = \pi_{a,i}^* - \frac{f\left(\pi_{a,i}^*\right)}{f'(\pi_{a,i}^*)}\frac{\partial^2\Omega}{\partial u^2} \qquad (12)$$

As follows, the distribution of π_a^* is simulated when $\alpha = 0.1, \beta = 0.2$. To this end, first, 1000 samples are generated of beta distribution with parameters β, α. Then, for each samples, the Newon-Raphson method is applied to find the converged $\pi_{a,i}^*, i \to \infty$ here, $\pi_{a,i}^*, i = 50$ is considered as a sample of π_a^*. The following Figure gives the histogram of π_a^*.

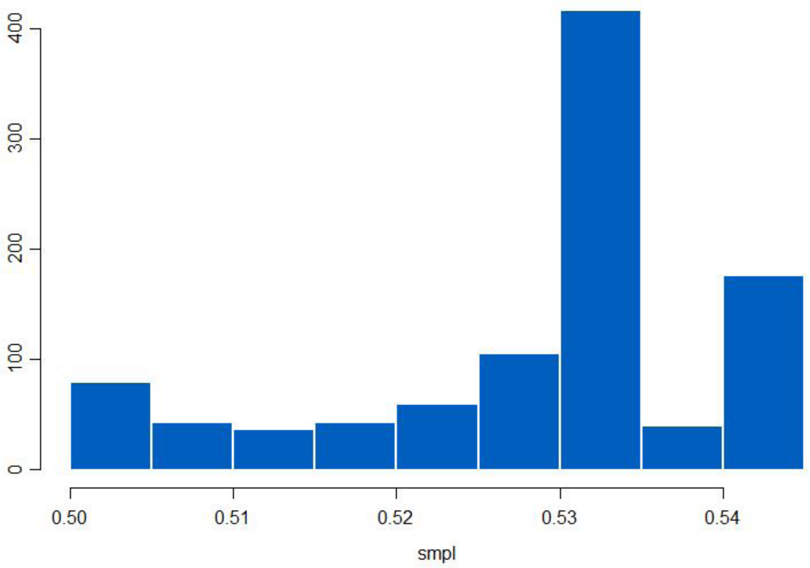

Figure 5: Empirical density of π_a^*: Bayesian equilibrium

The mean and standard deviation of π_a^* are 0.5277 and 0.0114. Thus, the mean of π_b^* are $\frac{0.2}{0.1+0.2} - 0.5277 = \frac{2}{3} - 0.5277 = 0.13897$ and assuming independence of π_a^* and π_b^*, then

$var(\Delta) = 0.175 = var\left(\pi_b^*\right) + 0.00012996$ .Therefore, standard deviation of π_b^* is 0.418. Here, independence is a simplifier assumption which is assumed in the case of random Δ. When, this parameter is fixed π_a^* and π_b^* are linear dependent, however, in the case of random Δ, as a simplifier assumption, they can be considered to be independent. Again, it is seen the Nash equilibrium occurs at 0.53 similar to methods *(a)* and *(d)* of section 2.2. The following Table gives the mean (left number in the parenthesis) and standard deviation (right number) of π_a^* for various values of α, β.

α / β	**0.1**	**0.5**	**0.9**	**1.5**
0.1	0.531,0.012	0.527,0.011	0.526,0.009	0.526,0.009
0.5	0.533,0.014	0.521,0.014	0.516,0.012	0.515,0.011
0.9	0.534,0.015	0.522,0.015	0.512,0.011	0.511,0.009
1.5	0.535,0.014	0.519,0.015	0.514,0.013	0.508,0.009

Table 5: Mean and standard deviations of Bayesian π_a^*

In almost all cells, the standard deviation is too small which indicates the accuracy of estimation and the mean lies in a range of 0.51 to 0.54 which is very similar to results of methods *(a)* and *(d)* of section 2.2.

Conclusions

The inter-correlation between two objects in a sample space (here, two candidates) causes spurious interpretation of selecting probability of a specified candidate based on observed probabilities. Indeed, actual probabilities are smaller than observed ones. Although, observed winner candidate and actual candidate are the same, however, with a larger spurious probability. One main practical consequences of observing large spurious probabilities is that the winner candidate, for example candidate a , thinks has $p_{a|\sim b}$ percent advocators while it is a spurious percent and it is far from actual hidden probability p_a . After selection, only p_a percents of society support the winner candidate and there is a high probability that his/her programs fail. Also, this type of inter-correlation causes a coordination game between two main candidates which leads to probability of selecting 0.51 to 0.65, based on various equilibriums and learning methods.

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